ANALYTIC SUBORDINATION FOR FREE COMPRESSION (PRELIMINARY VERSION)

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Abstract. We extend the free difference quotient coalgebra approach to analytic subordination to the case of a free compression in free probability.

1. Introduction

If μ, ν are Borel probability measures on $\mathbb{R}$, let $\mu \boxplus \nu$ denote their free additive convolution. The Cauchy transform $G_{\mu\boxplus\nu}$ is analytically subordinate to G_μ in the upper half plane, i.e. there is an analytic function $f : \mathbb{H}_+(\mathbb{C}) \to \mathbb{H}_+(\mathbb{C})$ such that $G_{\mu\boxplus\nu}(z) = G_\mu(f(z))$. This result is the main tool in proving regularity properties of free convolution, and was first proved by D. Voiculescu in [Voi93] under an easily removed genericity condition. Using combinatorial methods, P. Biane showed in [Bia98] that the subordination is an operator valued phenomenon. Namely if X, Y are self-adjoint and free random variables in a von Neumann algebra with faithful normal trace state, then the functions $(X - zI)^{-1}$ and $E_{W^*(X)}((X + Y) - zI)^{-1}$ satisfy an analytic subordination relation in the upper half plane. In [Voi00], it was shown that the subordination is due to a certain conditional expectation which is a coalgebra morphism between the free difference quotient coalgebras of ∂_{X+Y} and ∂_X. This approach extends to the B-valued case, i.e. when X and Y are self-adjoint elements in a von Neumann algebra with faithful normal trace state which are B-free, where $1 \in B$ is a W*-subalgebra.

In [AN96], A. Nica and R. Speicher showed that for any Borel probability measure μ on $\mathbb{R}$, there is a partially defined continuous free additive convolution semi-group starting at μ, i.e. a continuous family $\{\mu_t : t \geq 1\}$ such that $\mu = \mu_1$, $\mu_{s+t} = \mu_s \boxplus \mu_t$. In [BB04], S. T. Belinschi and H. Bercovici showed that the analytic subordination for $\mu^{\boxplus n}$ extends to μ_t. This can be used to prove certain regularity results for the free additive convolution semigroup. Here we present a proof of this result following the free difference quotient coalgebra approach, which allows a B-valued extension.

If X is a self-adjoint element in a W*-probability space (M, τ) with distribution μ, and p is a projection in M free from X with $\tau(p) = t^{-1}$, then μ_t is the distribution of $tpXp$ in $(pMp, t\tau|_{pMp})$. Here we show that a certain rescaled conditional expectation is a coalgebra morphism between the free difference quotient coalgebras of ∂_{tpXp} and ∂_X. We then follow the approach of ([Voi00]) to establish the subordination result.

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2. Preliminaries

2.1. Free difference quotient derivation

If B is a unital algebra over $\mathbb{C}$ and X is algebraically free from B, the free difference quotient is the derivation

$$\partial_{X:B} : B\langle X\rangle \to B\langle X\rangle \otimes B\langle X\rangle$$

which takes B to 0 and X to $1 \otimes 1$. $\partial_{X:B}$ is a coassociative comultiplication, i.e.

$$(\partial_{X:B} \otimes \mathrm{id}) \circ \partial_{X:B} = (\mathrm{id} \otimes \partial_{X:B}) \circ \partial_{X:B}$$

2.2. Corepresentations of derivation-comultiplications.

Suppose A is a unital algebra over $\mathbb{C}$ and that $\partial : A \to A \otimes A$ is a coassociative comultiplication which is a derivation with respect to obvious A-bimodule structure on $A \otimes A$. A corepresentation of (A, ∂) is a $n \times n$

matrix $(a_{ij})_{1\leq i,j\leq n}$ with entries in A such that

$$\partial a_{ij} = \sum_{1\leq k\leq n} a_{ik}\otimes a_{kj}$$

We recall the following characterization of invertible corepresentations from ([Voi00, Proposition 1.4]).

Proposition. *Let (A,∂) be as above, and suppose that $X\in A$ is such that $\partial(X)=1\otimes 1$. If $\alpha=(a_{ij})_{1\leq i,j\leq n}$ is a corepresentation of (A,∂), such that α is invertible in $\mathfrak{M}_n(A)$, then*

$$\alpha = ((n_{ij}-X\delta_{ij})_{1\leq i,j\leq n})^{-1}$$

for some $n_{ij}\in N=\operatorname{Ker}\partial$. Conversely, if $n_{ij}\in N=\operatorname{Ker}\partial$ are such that the matrix $\beta=(n_{ij}-X\delta_{ij})_{1\leq i,j\leq n}$ is invertible in $\mathfrak{M}_n(A)$, then $\alpha=\beta^{-1}$ is a corepresentation of (A,∂).

□

2.3. Conjugate variables

If M is a von Neumann algebra with faithful normal trace state τ, $1\in B\subset M$ is a W*-subalgebra, and $X=X^*\in M$ is algebraically free from B, then the conjugate variable $\mathscr{J}(X:B)$ is defined as the unique (if it exists) element in $L^1(W^*(B\langle X\rangle))$ such that

$$\tau(\mathscr{J}(X:B)m) = (\tau\otimes\tau)(\partial_{X:B}m) \quad m\in B\langle X\rangle$$

If $|\mathscr{J}(X:B)|_2<\infty$, then viewing $\partial_{X:B}$ as a densely defined unbounded operator from $L^2(W^*(B\langle X\rangle))\to L^2(W^*(B\langle X\rangle))\otimes L^2(W^*(B\langle X\rangle))$ we have $1\otimes 1\in\mathfrak{D}(\partial^*_{X:B})$, $\mathscr{J}(X:B)=\partial^*_{X:B}(1\otimes 1)$ and $\partial_{X:B}$ is closable, in particular it is closable in $\|\ \|$. (See [Voi98]).

2.4. Noncommutative power series

If K is a C*-algebra, A is a C*-subalgebra and $R>0$ then $A_R\{t\}$ (see [Voi98]) will denote the completion of the ring $A\langle t\rangle$ of noncommutative polynomials with coefficients in A with respect to the norm $|\ |_R$ defined by

$$|P|_R = \inf\sum_{k\in\mathbb{N}}\left\|a_1^{(k)}\right\|\cdots\left\|a_{n(k)}^{(k)}\right\|R^{n(k)-1}$$

where the infimum is taken over all representations of a noncommutative polynomial $P\in A\langle t\rangle$ as a sum with finite support of the form

$$P(t)=\sum_{k\in\mathbb{N}} a_1^{(k)}ta_2^{(k)}t\cdots a_{n(k)}^{(k)}$$

If $X\in K$ and $\|X\|<R$, then $f(X)$ is well defined for any $f(t)\in A_R\{t\}$ and $\|f(X)\|\leq|f|_R$.

2.5. Half-planes of a C*-algebra

If K is a C*-algebra, we define the upper and lower half-planes in K by

$$\mathbb{H}_+(K)=\{T\in K|\mathrm{Im}\ T\geq\epsilon 1 \text{ for some } \epsilon>0\}$$
$$\mathbb{H}_-(K)=\{T\in K|\mathrm{Im}\ T\leq-\epsilon 1 \text{ for some } \epsilon>0\}$$

If $T\in\mathbb{H}_+(K)$, then T is invertible, and

$$\left\|T^{-1}\right\|\leq\epsilon^{-1} \qquad -(\epsilon+\epsilon^{-1}\|T\|^2)^{-1}\geq\mathrm{Im}\ T^{-1}$$

In particular, $T^{-1}\in\mathbb{H}_-(K)$. (See [Voi00, 3.6]).

By $\Delta_+\mathfrak{M}_n(K)$ we will denote the set of matrices $\kappa=(k_{ij})_{1\leq i,j\leq n}\in\mathfrak{M}_n(K)$ such that $k_{ii}\in\mathbb{H}_+(K), 1\leq i\leq n$, and $k_{ij}=0$ for $i<j$. So $\Delta_+\mathfrak{M}_n(K)$ is the set of lower triangular matrices with diagonal entries in $\mathbb{H}_+(K)$. Let $\Delta_-\mathfrak{M}_n(K)$ denote the lower triangular matrices with diagonal entries in $\mathbb{H}_-(K)$. Note that if $\kappa\in\Delta_\pm\mathfrak{M}_n(K)$ then $\kappa^{-1}\in\Delta_\mp\mathfrak{M}_n(K)$ and $(\kappa^{-1})_{ii}=(\kappa_{ii})^{-1}$ for $1\leq i\leq n$.

3. The coalgebra morphism associated to free compression

3.1. In this section we study a certain rescaled conditional expectation, which for a free compression gives a coalgebra morphism between free difference quotient coalgebras. The framework is (M, τ), a von Neumann algebra with a faithful normal trace state. If A, B are subalgebras in M, $A \vee B$ will denote the subalgebra generated (algebraically) by $A \cup B$. If $1 \in A \subset M$ is a $*$-subalgebra, $E_A^{(M)}$ will denote the unique conditional expectation of M onto $W^*(A)$ which preserves τ. If $p \in M$ is a projection in M, τ_p will denote the faithful normal trace state on pMp given by $\tau_p = \tau(p)^{-1}\tau|_{pMp}$.

Lemma 3.2. *Suppose that $1 \in B \subset M$ is a $*$-subalgebra, $X = X^* \in M$ and that $p \in M$ is a projection such that p commutes with B and X is algebraically free from $B[p]$. Let α denote $\tau(p)$, and put $X_p = \alpha^{-1}pXp$, which we consider as a Bp-valued random variable in pMp. Define $\psi : pMp \to M$ by $\psi(pmp) = \alpha^{-1}pmp$. Then $\psi(Bp\langle X_p\rangle) \subset B\langle p, X\rangle$ and*

$$(\psi \otimes \psi) \circ \partial_{X_p:Bp} = \partial_{X:B[p]} \circ \psi|_{Bp\langle X_p\rangle}$$

i.e., $\psi|_{B_p\langle X_p\rangle}$ is a coalgebra morphism for the comultiplications $\partial_{X_p:Bp}$ and $\partial_{X:B[p]}$.

Proof. Clearly $\psi(Bp\langle X_p\rangle) \subset B\langle p, X\rangle$, we must show that ψ is comultiplicative. Both sides of the above equation are derivations from $Bp\langle X_p\rangle$ into $M \otimes M$ with respect to the natural $Bp\langle X_p\rangle$ bimodule structure on $M \otimes M$. It is clear that Bp is in the kernel of both derivations, we need only compare them on X_p. We have

$$\partial_{X:B[p]} \circ \psi(X_p) = \alpha^{-2}\partial_{X:B[p]}(pXp) = \alpha^{-2}p \otimes p = (\psi \otimes \psi)(p \otimes p) = (\psi \otimes \psi) \circ \partial_{X_p:Bp}(X_p)$$

□

3.3. Certain conditional expectations behave well with respect to freeness and derivations, which allows us to extend the coalgebra morphism ψ to a rescaled conditional expectation. We will need the following result from ([Voi00, Lemma 2.2]).

Lemma. *Let $1 \in B$ be a W^*-subalgebra, and let $1 \in A, 1 \in C$ be $*$-subalgebras in (M, τ). Assume A and C are B-free in (M, E_B). Let $D : A \vee B \vee C \to (A \vee B \vee C) \otimes (A \vee B \vee C)$ be a derivation such that $D(B \vee C) = 0$ and $D(A \vee B) \subset (A \vee B) \otimes (A \vee B)$. Then*

$$(E_{A\vee B} \otimes E_{A\vee B}) \circ D = D \circ E_{A\vee B}|_{A\vee B\vee C}$$

□

Proposition 3.4. *Suppose that $1 \in B \subset M$ is a W*-subalgebra, $X = X^* \in M$ and that $p \in M$ is a projection such that p is B-free with X, p commutes with B and X is algebraically free from $B[p]$. Let α denote $\tau(p)$, and put $X_p = \alpha^{-1}pXp$. Define $\Psi : pMp \to M$ by $\Psi = E_{B\langle X\rangle}^{(M)} \circ \psi$. Then*

$$(\Psi \otimes \Psi) \circ \partial_{X_p:Bp} = \partial_{X:B} \circ \Psi|_{Bp\langle X_p\rangle}$$

Proof. Since X and p are B-free in M, $E_{B\langle X\rangle}^{(M)} B[X, p] \subset B\langle X\rangle$ so that

$$\Psi(Bp[X_p]) \subset B\langle X\rangle$$

By the previous lemma applied to $A = \mathbb{C}[X], B = B, C = \mathbb{C}[p], D = \partial_{X:B[p]}$ we have

$$\left(E_{B\langle X\rangle}^{(M)} \otimes E_{B\langle X\rangle}^{(M)}\right) \circ \partial_{X:B[p]} = \partial_{X:B} \circ E_{B\langle X\rangle]}^{(M)}\Big|_{B\langle X,p\rangle}$$

The result then follows from composing both sides with $\psi|_{Bp[X_p]}$ and applying Lemma 3.2. □

3.5. To attach probabilistic meaning to the map Ψ, it should be unital and preserve trace and expectation onto B. These properties require the additional assumption that p is independent from B with respect to τ.

Proposition. *Let M, B, X, p, Ψ as above and suppose, in addition to the previous hypotheses, that p is independent from B with respect to τ. Then $\Psi(bp) = b$ for $b \in B$, in particular Ψ is unital. Furthermore, Ψ preserves trace and expectation onto B, i.e.*

$$\tau \circ \Psi = \tau_p$$

$$\Psi \circ E_{Bp}^{(pMp)} = E_B^{(M)} \circ \Psi$$

Proof. First remark that independence implies $E_B^{(M)}(p) = \alpha$. Since X and p are B-free,

$$E_{B\langle X\rangle}^{(M)}(p) = E_B^{(M)}(p) = \alpha$$

Therefore, for $b \in B$ we have

$$\Psi(bp) = \alpha^{-1} E_{B\langle X\rangle}^{(M)}(bp) = \alpha^{-1} b E_{B\langle X\rangle}^{(M)}(p) = b$$

Next observe that

$$\begin{aligned}\tau\left(\Psi(pmp)\right) &= \alpha^{-1}\tau\left(E_{B\langle X\rangle}^{(M)}(pmp)\right)\\ &= \alpha^{-1}\tau(pmp)\\ &= \tau_p(pmp)\end{aligned}$$

so that Ψ preserves trace. Next we claim that

$$E_{Bp}^{(pMp)}(pmp) = \alpha^{-1} E_B^{(M)}(pmp)p$$

First observe that the right hand side is a conditional expectation from pMp onto $W^*(Bp)$. Since $E_{Bp}^{(pMp)}$ is the unique such conditional expectation which preserves τ_p, it remains only to show that this map is trace preserving. We have

$$\tau_p\left(\alpha^{-1}E_B^{(M)}(pmp)p\right) = \alpha^{-2}\tau\left(E_B^{(M)}(pmp)p\right) = \alpha^{-1}\tau(pmp) = \tau_p(pmp)$$

which proves the claim. We then have

$$\begin{aligned}\left(\Psi\circ E_{Bp}^{(pMp)}\right)(pmp) &= \Psi\left(\alpha^{-1}E_B^{(M)}(pmp)p\right)\\ &= \alpha^{-2}E_{B\langle X\rangle}^{(M)}\left(E_B^{(M)}(pmp)p\right)\\ &= E_B^{(M)}\left(\alpha^{-1}E_{B\langle X\rangle}^{(M)}(pmp)\right)\\ &= \left(E_B^{(M)}\circ\Psi\right)(pmp)\end{aligned}$$

So that Ψ preserves expectation onto B.

□

3.6. If $X = X^*, Y = Y^* \in M$ are B-free, where $1 \in B \subset M$ is a W*-subalgebra, then if $\mathscr{J}(X : B)$ exists so does $\mathscr{J}(X+Y : B)$ and is obtained from a conditional expectation. This is also true for a free compression:

Proposition. *Suppose that $1 \in B \subset M$ is a W*-subalgebra, $X = X^* \in M$ and that $p \in M$ is a projection such that p commutes with B and X is algebraically free from $B[p]$. Let α denote $\tau(p)$, and put $X_p = \alpha^{-1}pXp$. Assume that p and B are independent, and that X and p are B-freely independent. If $\mathscr{J}(X : B)$ exists, then $\mathscr{J}(X_p : Bp)$ exists and is given by*

$$E_{Bp\langle X_p\rangle}^{(pMp)}(p\mathscr{J}(X:B)p)$$

Proof. Let Ψ be as above, then for $pmp \in Bp\langle X_p\rangle$ we have

$$\begin{aligned}(\tau_p\otimes\tau_p)(\partial_{X_p:Bp}(pmp)) &= (\tau\otimes\tau)(\partial_{X:B}\Psi(pmp))\\ &= \alpha^{-1}\tau\left(\mathscr{J}(X:B)E_{B\langle X\rangle}^{(M)}(pmp)\right)\\ &= \alpha^{-1}\tau(\mathscr{J}(X:B)pmp)\\ &= \tau_p((p\mathscr{J}(X:B)p)pmp)\\ &= \tau_p\left(E_{Bp\langle X_p\rangle}^{(pMp)}(p\mathscr{J}(X:B)p)pmp\right)\end{aligned}$$

□

4. Completely positive morphisms between free difference quotient coalgebras

In this section we will prove the analytic subordination result for a free compression. We will follow Voiculescu's approach in ([Voi00, Section 3]).

4.1. We begin with a standard result on unbounded derivations on C*-algebras ([Voi00],[BR79])

Lemma. *Let K, L be unital C*-algebras, let $\varphi_1, \varphi_2 : K \to L$ be unital $*$-homorphisms, let $1 \in A \subset K$ be a unital $*$-subalgebra, and let $D : A \to L$ be a closable derivation with respect to the A-bimodule structure on L defined by φ_1, φ_2. The closure $\overline{D}$ is then a derivation, and the domain of definition $\mathfrak{D}(\overline{D})$ is a subalgebra. Moreover, if $a \in A$ is invertible in K, then $a^{-1} \in \mathfrak{D}(\overline{D})$ and*

$$\overline{D}(a^{-1}) = -\varphi_1(a^{-1})D(a)\varphi_2(a^{-1})$$

□

We will now restate two lemmas from [Voi00, Section 3] in the C*-context, since the proofs carry over directly we will omit them.

Lemma 4.2. *Let K be a unital C*-algebra and $1 \in A \subset K$ a C*-subalgebra. Suppose $X = X^* \in K$ is algebraically free from A, $\|X\| \leq R$ and $\partial_{X:A}$ is closable. If $f \in A_R\{t\}$ then $f(X) \in \mathfrak{D}(\overline{\partial}_{X:A})$. Moreover, if $\overline{\partial}_{X:A} f(X) = 0$, then $f(X) \in A$.*

□

Lemma 4.3. *Let K, A and X as above, $f \in A\langle t\rangle$, and let $P = f(X)$. Then*

$$|f|_R \leq \sum_{p\geq 0} \left\| \partial^{(p)}_{X:A} P \right\|^{\widehat{}}_{(p+1)} (\|X\| + R)^p$$

where $\| \ \|^{\widehat{}}_{(s)}$ is the norm on the s-fold projective tensor product $K^{\widehat{\otimes} s}$.

□

We are now prepared to prove a subordination result for completely positive coalgebra morphisms between free difference quotient coalgebras. The proof follows ([Voi00, Proposition 3.7]).

Theorem 4.4. *Let K and L be unital C*-algebras, and $1 \in A \subset K$, $1 \in B \subset L$ C*-subalgebras. Let $X = X^* \in K$ algebraically free from A, $Y = Y^* \in L$ algebraically free from B. Suppose $\Psi : L \to K$ is a unital, completely positive linear map such that $\Psi(B\langle Y\rangle) \subset A\langle X\rangle$ and*

$$(\Psi \otimes \Psi) \circ \partial_{Y:B} = \partial_{X:A} \circ \Psi|_{B\langle Y\rangle}$$

Suppose also that $\partial_{X:A}$ and $\partial_{Y:B}$ are closable. Then there is a holomorphic map $F_n : \mathbb{H}_+(\mathfrak{M}_n(B)) \to \mathbb{H}_+(\mathfrak{M}_n(A))$ such that

$$\mathfrak{M}_n(\Psi)\left((Y \otimes I_n - \beta)^{-1}\right) = (X \otimes I_n - F_n(\beta))^{-1}$$

for $\beta \in \mathbb{H}_+(\mathfrak{M}_n(B))$.

Proof. By replacing (K, L, A, B, X, Y, Ψ) with $(\mathfrak{M}_n(K), \mathfrak{M}_n(L), \mathfrak{M}_n(A), \mathfrak{M}_n(B), X \otimes I_n, Y \otimes I_n, \mathfrak{M}_n(\Psi))$, we may assume without loss of generality that $n = 1$. Let $\overline{\partial}_{X:A}$ and $\overline{\partial}_{Y:B}$ denote the closures of $\partial_{X:A}$ and $\partial_{Y:B}$. We have $\Psi(\mathfrak{D}(\overline{\partial}_{Y:B})) \subset \mathfrak{D}(\overline{\partial}_{X:A})$ and

$$(\Psi \otimes \Psi) \circ \overline{\partial}_{Y:B} = \overline{\partial}_{X:A} \circ \Psi|_{B\langle Y\rangle}$$

For $\beta \in \mathbb{H}_+(B)$,

$$(\beta - Y)^{-1} \in \mathfrak{D}(\overline{\partial}_{Y:B})$$

by Proposition 4.1, and so $(\beta - Y)^{-1}$ is a corepresentation of the coalgebra $(\mathfrak{D}(\overline{\partial}_{Y:B}), \overline{\partial}_{Y:B})$ by (2.2). Therefore $\gamma = \Psi((\beta - Y)^{-1})$ is a corepresentation of $(\mathfrak{D}(\overline{\partial}_{X:A}), \overline{\partial}_{X:A})$. Since $(\beta - Y)^{-1} \in \mathbb{H}_-(L)$ and Ψ is positive and unital, we have $\gamma \in \mathbb{H}_-(K)$. In particular γ is invertible. Note that since

$$\partial_{X:A}(a^*) = \sigma_{12}((\partial_{X:A}(a))^*)$$

where σ_{12} is the automorphism of $M \otimes M$ defined by $\sigma_{12}(m_1 \otimes m_2) = m_2 \otimes m_1$, $\mathfrak{D}(\overline{\partial}_{X:A})$ is a $*$-algebra. Hence $\gamma^{-1} \in \mathfrak{D}(\overline{\partial}_{X:A})$ by Proposition 4.1. By (2.2),

$$\gamma^{-1} = \eta - X$$

for some $\eta \in \operatorname{Ker} \overline{\partial}_{X:A}$. Since $\gamma^{-1} \in \mathbb{H}_+(K)$, we have $\eta \in \mathbb{H}_+(K)$. Clearly the map taking $\beta \in \mathbb{H}_+(B)$ to $\eta \in \mathbb{H}_+(K)$ is a holomorphic map, it remains only to show that $\eta \in A$. By analytic continuation, it suffices to show this for β in an open subset of $\mathbb{H}_+(B)$.

Let $\rho = 6(\|X\| + \|Y\| + 1)$, and put

$$\omega = \{\beta \in B | \, \|i\rho - \beta\| < 1\} \subset \mathbb{H}_+(B)$$

If $\beta \in \omega$, then

$$(\beta - Y)^{-1} = (i\rho(1-\Gamma))^{-1} = (i\rho)^{-1} \sum_{m \geq 0} \Gamma^m$$

where

$$\Gamma = (i\rho)^{-1}(i\rho - \beta + Y)$$

Note that $\|\Gamma\| < 1/6$.

Let $\| \, \|^{\widehat{}}_{(p)}$ denote the projective tensor product norm on $K^{\widehat{\otimes} p}$. Define $\varphi_j : K \to K^{\widehat{\otimes}(p+1)}$ by $\varphi_j(k) = 1^{\otimes(j-1)} \otimes k \otimes 1^{\otimes(p+1)-j}$. Then since $\partial_{Y:B}\Gamma = (i\rho)^{-1} 1 \otimes 1$, it follows easily that

$$\partial^{(p)}_{Y:B}\Gamma^m = \sum_{\substack{m_1 \geq 0, \ldots, m_{p+1} \geq 0 \\ m_1 + \cdots m_{p+1} = m-p}} (i\rho)^{-p} \varphi_1(\Gamma^{m_1}) \cdots \varphi_{p+1}(\Gamma^{m_{p+1}})$$

From this it follows that

$$\left\| \partial^{(p)}_{Y:B}\Gamma^m \right\|^{\widehat{}}_{(p+1)} < \rho^{-p} 6^{-(m-p)} \frac{m!}{p!(m-p)!}$$

if $m \geq p$, while if $m < p$ then

$$\partial^{(p)}_{Y:B}\Gamma^m = 0$$

Let $P_m = \Psi(\Gamma^m)$. Then $P_m \in A\langle X \rangle$ and

$$\partial^{(p)}_{X:A} P_m = \Psi^{\otimes(p+1)}(\partial^{(p)}_{Y:B}\Gamma^m)$$

Hence

$$\left\| \partial^{(p)}_{X:A} P_m \right\|^{\widehat{}}_{(p+1)} \leq \rho^{-p} 6^{-(m-p)} \frac{m!}{p!(m-p)!}$$

if $m \geq p$ and is zero if $m < p$. Let $h_m \in A\langle t \rangle$ so that $P_m = h_m(X)$. By Lemma 4.3,

$$\begin{aligned} |h_m|_r &\leq \sum_{p \geq 0} \left\| \partial^{(p)}_{X:A} P_m \right\|^{\widehat{}}_{(p+1)} (\|X\| + r)^p \\ &< \sum_{0 \leq p \leq m} \rho^{-p} 6^{-(m-p)} (\|X\| + r)^p \frac{m!}{p!(m-p)!} \\ &= (\rho^{-1}(\|X\| + r) + 6^{-1})^m \end{aligned}$$

Let $r = \|X\| + 1$, so that

$$|h_m|_r < (1/2)^m$$

if $m \geq 1$. Then $h = \sum_{m \geq 1} h_m \in A_r\{t\}$ and $|h|_r < 1$. It follows that $1 + h$ is invertible in $A_r\{t\}$. We then have

$$\begin{aligned} \eta - X &= (\Psi(\beta - Y)^{-1})^{-1} \\ &= (i\rho) \left(1 + \sum_{k \geq 1} P_k \right)^{-1} \\ &= (i\rho)(1+h)^{-1}(X) \end{aligned}$$

Hence $\overline{\partial}_{X:A}\eta = 0$ and $\eta = g(X)$ where $g = t + (i\rho)(1+h)^{-1} \in A_r\{t\}$ and $r = \|X\| + 1$. By Lemma 4.2, $\eta \in A$. $\square$

Corollary 4.5. *Let (M,τ) be a von Neumann algebra with faithful normal trace state, and $1 \in B \subset M$ a W^*-subalgebra. Suppose $X = X^* \in M$ and that $p \in M$ is a projection which is B-free with X and such that p is independent from B with respect to τ. Let α denote $\tau(p)$, and put $X_p = \alpha^{-1}pXp$. Assume that $|\mathscr{J}(X:B)|_2 < \infty$. Then there is an analytic function $F_n : \mathbb{H}_+(\mathfrak{M}_n(B)) \to \mathbb{H}_+(\mathfrak{M}_n(B))$ such that*

$$\alpha^{-1}E^{(\mathfrak{M}_n(M))}_{\mathfrak{M}_n(B\langle X\rangle)}\left(X_p \otimes I_n - \beta(p\otimes I_n)\right)^{-1} = \left(X\otimes I_n - F_n(\beta)\right)^{-1}$$

for $\beta \in \mathbb{H}_+(\mathfrak{M}_n(B))$.

Proof. By Proposition 3.6, also $|\mathscr{J}(X_p : Bp)|_2 < \infty$, hence $\partial_{X:B}$ and $\partial_{X_p:Bp}$ are closable in norm. By Proposition 3.4, Theorem 4.4 applies to $K = M$, $L = pMp$, $A = B$, $B = Bp$, $X = X$, $Y = X_p$, $\Psi = \Psi$ which gives the result. ☐

4.6. In B-valued free probability, it is useful also to consider matricial resolvents $(X\otimes I_n - \beta)^{-1}$ where $\beta \in \Delta_+\mathfrak{M}_n(B)$ (see [Voi86]). The subordination extends also to these resolvents.

Corollary. *Let K and L be unital C^*-algebras, and $1 \in A \subset K$, $1 \in B \subset L$ C^*-subalgebras. Let $X = X^* \in K$ algebraically free from A, $Y = Y^* \in L$ algebraically free from B. Suppose $\Psi : L \to K$ is a unital, completely positive linear map such that $\Psi(B\langle Y\rangle) \subset A\langle X\rangle$ and*

$$(\Psi\otimes\Psi)\circ\partial_{Y:B} = \partial_{X:A}\circ\Psi|_{B\langle Y\rangle}$$

Suppose also that $\partial_{X:A}$ and $\partial_{Y:B}$ are closable. Then there is a holomorphic map $\Phi_n : \Delta_+\mathfrak{M}_n(B) \to \Delta_+\mathfrak{M}_n(A)$ such that

$$\mathfrak{M}_n(\Psi)\left((Y\otimes I_n - \beta)^{-1}\right) = \left(X\otimes I_n - \Phi_n(\beta)\right)^{-1}$$

for $\beta \in \Delta_+\mathfrak{M}_n(B)$.

Proof. Let $\beta = (b_{ij})_{1\le i,j\le n}$, and $\gamma = (\gamma_{ij})_{1\le i,j\le n}$ where

$$\gamma = \mathfrak{M}_n(\Psi)\left((Y\otimes I_n - \beta)^{-1}\right)$$

Then since $\beta \in \Delta_+\mathfrak{M}_n(B)$ we have $\gamma \in \Delta_+\mathfrak{M}_n(A\langle X\rangle)$ and

$$\gamma_{ii} = \Psi\left((Y - b_{ii})^{-1}\right)$$

By Theorem 4.4, $\gamma_{ii} = (X - \eta_{ii})^{-1}$ for some $\eta_{ii} \in \mathbb{H}_+(A)$. Since $\gamma \in \Delta_+\mathfrak{M}_n(A\langle X\rangle)$, $\gamma^{-1} \in \Delta_-\mathfrak{M}_n(A\langle X\rangle)$ and

$$(\gamma^{-1})_{ii} = (\gamma_{ii})^{-1} = X - \eta_{ii}$$

so that $\gamma^{-1} - X\otimes I_n = \eta$ for some $\eta \in \Delta_-\mathfrak{M}_n(A)$. The analytic dependence on β is clear, so this completes the proof. ☐

Corollary 4.7. *Let (M,τ) be a von Neumann algebra with faithful normal trace state, and $1 \in B \subset M$ a W^*-subalgebra. Suppose $X = X^* \in M$ and that $p \in M$ is a projection which is B-free with X and such that p is independent from B with respect to τ. Let α denote $\tau(p)$, and put $X_p = \alpha^{-1}pXp$. Assume that $|\mathscr{J}(X:B)|_2 < \infty$. Then there is an analytic function $\Phi_n : \Delta_+\mathfrak{M}_n(B) \to \Delta_+\mathfrak{M}_n(B)$ such that*

$$\alpha^{-1}E^{(\mathfrak{M}_n(M))}_{\mathfrak{M}_n(B\langle X\rangle)}\left(X_p \otimes I_n - \beta(p\otimes I_n)\right)^{-1} = \left(X\otimes I_n - F_n(\beta)\right)^{-1}$$

for $\beta \in \Delta_+\mathfrak{M}_n(B)$. ☐

5. Free Markovianity for Free Compression

We can now remove the condition on $|\mathscr{J}(X:B)|_2 < \infty$ from Corollary 4.5. The key tool is the following "Free Markovianity" property of free compression.

Proposition 5.1. *Suppose that $1 \in B \subset M$ is a W^*-subalgebra, $X = X^* \in M$ and that $p \in M$ is a projection such that p commutes with B, and X and p are B-freely independent. Let $Y = Y^* \in M$ be B-free from $B\langle X, p\rangle$. Then*

$$E^{(M)}_{B\langle X\rangle}E^{(M)}_{B\langle X+Y\rangle}E^{(pMp)}_{Bp\langle p(X+Y)p\rangle} = E^{(M)}_{B\langle X\rangle}E^{(pMp)}_{Bp\langle p(X+Y)p\rangle}$$

Proof. Apply [Voi99, Lemma 3.3] to

$$\begin{aligned} D &= B, & B &= W^*(B\langle X+Y\rangle) \\ A_1 &= W^*(B\langle X,Y\rangle) \\ A &= W^*(B\langle X\rangle), & \Omega &= \{p\} \\ C &= W^*(B\langle X+Y,p\rangle) \end{aligned}$$

to conclude that $W^*(B\langle X\rangle)$, $W^*(B\langle X+Y\rangle)$, $W^*(B\langle X+Y,p\rangle)$ is freely Markovian. By [Voi99, Lemma 3.7]

$$E^{(M)}_{B\langle X\rangle}E^{(M)}_{B\langle X+Y\rangle}E^{(M)}_{B\langle X+Y,p\rangle} = E^{(M)}_{B\langle X\rangle}E^{(M)}_{B\langle X+Y,p\rangle}$$

Since $Bp\langle p(X+Y)p\rangle \subset B\langle X+Y,p\rangle$,

$$E^{(M)}_{B\langle X+Y,p\rangle}E^{(pMp)}_{Bp\langle p(X+Y)p\rangle} = E^{(pMp)}_{Bp\langle p(X+Y)p\rangle}$$

from which the result follows. □

We are now prepared to remove the condition on $\mathscr{J}(X:B)$, this follows ([Voi00, Theorem 3.8]).

Theorem 5.2. *Let (M,τ) be a von Neumann algebra with faithful normal trace state, and $1 \in B \subset M$ a W^*-subalgebra. Suppose $X = X^* \in M$ and that $p \in M$ is a projection which is B-free with X and such that p is independent from B with respect to τ. Let α denote $\tau(p)$, and put $X_p = \alpha^{-1}pXp$. Then there is an analytic function $F_n : \mathbb{H}_+(\mathfrak{M}_n(B)) \to \mathbb{H}_+(\mathfrak{M}_n(B))$ such that*

$$\alpha^{-1}E^{(\mathfrak{M}_n(M))}_{\mathfrak{M}_n(B\langle X\rangle)}\left(X_p \otimes I_n - \beta(p\otimes I_n)\right)^{-1} = (X\otimes I_n - F_n(\beta))^{-1}$$

for $\beta \in \mathbb{H}_+(\mathfrak{M}_n(B))$.

Proof. The analytic dependence on β is clear, so we must prove that

$$\alpha^{-1}E^{(\mathfrak{M}_n(M))}_{\mathfrak{M}_n(B\langle X\rangle)}(X_p \otimes I_n - \beta(p\otimes I_n))^{-1} = (X\otimes I_n - \eta)^{-1}$$

for some $\eta \in \mathbb{H}_+(\mathfrak{M}_n(B))$. Let S be a $(0,1)$-semicircular element in (M,τ) which is freely independent from $B\langle X,p\rangle$. Then X, p and S are B-free ([Voi99, Lemma 3.3]). Also $X+\epsilon S$ and p are B-free and $|\mathscr{J}(X+\epsilon S:B)|_2 < \infty$ for $\epsilon > 0$ by [Voi98, Corollary 3.9]. So we can apply Corollary 4.5 to $B, X+\epsilon S, p$, it follows that there are $\eta(\epsilon) \in \mathbb{H}_+(\mathfrak{M}_n(B))$ for $0 < \epsilon \le 1$ such that

$$\alpha^{-1}E^{(\mathfrak{M}_n(M))}_{\mathfrak{M}_n(B\langle X+\epsilon S\rangle)}(\alpha^{-1}p(X+\epsilon S)p\otimes I_n - \beta(p\otimes I_n))^{-1} = ((X+\epsilon S)\otimes I_n - \eta(\epsilon))^{-1}$$

Then

$$\left\|((X+\epsilon S)\otimes I_n - \eta(\epsilon))^{-1}\right\| \le \left\|(\alpha^{-1}p(X+\epsilon S)p\otimes I_n - \beta(p\otimes I_n))^{-1}\right\| \le C_1$$

for some fixed constant C_1, and by (2.5) also

$$\operatorname{Im}\,((X+\epsilon S)\otimes I_n - \eta(\epsilon))^{-1} \ge C_2(1\otimes I_n)$$

for some constant $C_2 > 0$. By (2.5), it follows that

$$\|\eta(\epsilon)\| \le C_3, \qquad \operatorname{Im}\,\eta(\epsilon) \ge C_4(1\otimes I_n)$$

for some constants $C_3, C_4 > 0$. It follows that

$$\lim_{\epsilon\to 0}\left\|((X+\epsilon S)\otimes I_n - \eta(\epsilon))^{-1} - ((X\otimes I_n) - \eta(\epsilon))^{-1}\right\| = 0$$

and hence

$$\lim_{\epsilon\to 0}\left\|((X+\epsilon S)\otimes I_n - \eta(\epsilon))^{-1} - E^{\mathfrak{M}_n(M))}_{\mathfrak{M}_n(B\langle X\rangle)}((X+\epsilon S)\otimes I_n - \eta(\epsilon))^{-1}\right\| = 0$$

By the previous proposition,

$$E^{(M)}_{B\langle X\rangle}E^{(M)}_{B\langle X+\epsilon S\rangle}E^{(pMp)}_{Bp\langle\alpha^{-1}p(X+\epsilon S)p\rangle} = E^{(M)}_{B\langle X\rangle}E^{(pMp)}_{Bp\langle\alpha^{-1}p(X+\epsilon S)p\rangle}$$

so that

$$\lim_{\epsilon\to 0}\left\|((X+\epsilon S)\otimes I_n - \eta(\epsilon))^{-1} - \alpha^{-1}E^{(M_n(M))}_{\mathfrak{M}_n(B\langle X\rangle)}((\alpha^{-1}p(X+\epsilon S)p)\otimes I_n - \beta(p\otimes I_n))^{-1}\right\| = 0$$

But also

$$\lim_{\epsilon\to 0}\left\|((\alpha^{-1}p(X+\epsilon S)p)\otimes I_n - \beta(p\otimes I_n))^{-1} - (X_p\otimes I_n - \beta(p\otimes I_n))^{-1}\right\| = 0$$

Putting these equations together, we have that

$$\alpha^{-1}E^{(\mathfrak{M}_n(M))}_{\mathfrak{M}_n(B\langle X\rangle)}(X_p\otimes I_n-\beta(p\otimes I_n))^{-1}=\lim_{\epsilon\to 0}(X\otimes I_n-\eta(\epsilon))^{-1}$$

which implies that $\eta(\epsilon)$ converges in norm to some $\eta\in\mathbb{H}_+(\mathfrak{M}_n(B))$ with

$$\alpha^{-1}E^{(\mathfrak{M}_n(M))}_{\mathfrak{M}_n(B\langle X\rangle)}(X_p\otimes I_n-\beta(p\otimes I_n))^{-1}=(X\otimes I_n-\eta)^{-1}$$

□

5.3. The same proof as Corollary 4.6 can be used to extend Theorem 5.2 to resolvents in $\Delta_+\mathfrak{M}_n(B)$.

Corollary. *Let (M,τ) be a von Neumann algebra with faithful normal trace state, and $1\in B\subset M$ a W^*-subalgebra. Suppose $X=X^*\in M$ and that $p\in M$ is a projection which is B-free with X and such that p is independent from B with respect to τ. Let α denote $\tau(p)$, and put $X_p=\alpha^{-1}pXp$. Then there is an analytic function $\Phi_n:\Delta_+\mathfrak{M}_n(B)\to\Delta_+\mathfrak{M}_n(B)$ such that*

$$\alpha^{-1}E^{(\mathfrak{M}_n(M))}_{\mathfrak{M}_n(B\langle X\rangle)}\left(X_p\otimes I_n-\beta(p\otimes I_n)\right)^{-1}=\left(X\otimes I_n-\Phi_n(\beta)\right)^{-1}$$

for $\beta\in\Delta_+\mathfrak{M}_n(B)$. □

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